Integration by parts

by Clive Newstead

Integration by parts is a direct result of the product rule. The product rule says that

$$\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

If we integrate both sides of this with respect to x, we find

$$uv = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

And hence, rearranged in its usual form, we have

$$\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx \tag{1}$$

If there are limits involved, we have the similar result

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b \frac{du}{dx}v\,dx \tag{2}$$

When we integrate by parts, we choose what is going to be our u function (to be differentiated) and what is going to be our $\frac{dv}{dx}$ (to be integrated), find our values for $\frac{du}{dx}$ and v, and then substitute them into equation (1) or (2), depending on whether limits are involved or not.

Before I go into details, you might want to ask, why is integration by parts useful? Well, we often find ourselves needing to integrate something which looks fairly simple, but for which we don't have a standard integral result that we can just apply (e.g. $\ln x$). Sometimes we find that if what we are trying to integrate is the product of two functions (e.g. $x \sin x$), then differentiating one of them and integrating the other produces something that we know how to integrate.

There are a many different types of functions that integration by parts is useful for, and some of these can be classified into three categories:

1. Integrals of the form $\int x^n f(x)\,dx$
2. Integrals of the form $\int 1 \times f(x)\,dx$
3. Integrals of products of functions with cyclic derivatives (e.g. $\sin x$, $\cos x$, e^x, $\sinh x$)

It is with regard to these categories that I will structure the rest of this guide.

Integrals of the form $\int x^n f(x)\,dx$

Often integration by parts problems involve the integration of functions of the form $x^n f(x)$. The idea when doing this is to get rid of the x^n term completely by differentiating it n times, because each time you differentiate the power is reduced by 1. Usually, in exams, $n = 1$, and so we only need to use integration by parts once! Lucky for us. Rarely you might be asked to integrate the likes of $x^2 \sin x$, which uses integration by parts twice but such questions are a rarity.

What we do to integrate such problems is set $u = x^n$ and $\dfrac{dv}{dx} = f(x)$, so that $\dfrac{du}{dx} = nx^{n-1}$ and $v = F(x)$ (where $F(x) = \int f(x)\,dx$). Say, for example, that our integral is

$$\int x \sin x \, dx$$

Then

$$\begin{cases} u = x & \Rightarrow \frac{du}{dx} = 1 \\ \frac{dv}{dx} = \sin x & \Rightarrow v = -\cos x \end{cases}$$

And so substituting into equation (1) we get

$$\begin{aligned} \int x \sin x \, dx &= -x \cos x - \int (-\cos x)\,dx \\ &= -x\cos x + \sin x + C \end{aligned}$$

This was fairly simple to do. What if we had $x^2 \sin x$ instead of $x \sin x$? Well, we just do the same, except we need to apply integration by parts twice. First of all we have

$$\begin{cases} u_1 = x^2 & \Rightarrow \frac{du_1}{dx} = 2x \\ \frac{dv_1}{dx} = \sin x & \Rightarrow v_1 = -\cos x \end{cases}$$

This gives

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int (-2x \cos x)\,dx$$

Now we need to find what $\int (-2x \cos x)\,dx$ is; well, we need to use integration by parts again, with

$$\begin{cases} u_2 = -2x & \Rightarrow \frac{du_2}{dx} = -2 \\ \frac{dv_2}{dx} = \cos x & \Rightarrow v_2 = \sin x \end{cases}$$

Again substituting into equation (1) we obtain

$$\begin{aligned} \int x^2 \sin x \, dx &= -x^2 \cos x - \left[-2x \sin x - \int (-2\sin x)\,dx\right] \\ &= -x^2 \cos x - [-2x\sin x - 2\cos x + C] \\ &= -x^2 \cos x + 2x \sin x + 2\cos x + C \\ &= 2x \sin x + (2 - x^2)\cos x + C \end{aligned}$$

What I've done above is more or less a standard procedure for all integrals of functions which take the form $x^n f(x)$. It's worth giving it some practice because it can be easy to mess it up, especially with all the minus signs floating about!

Integrals of the form $\int 1 \times f(x)\,dx$

You're probably wondering why I've chosen to write $\int 1 \times f(x)\,dx$ rather than $\int f(x)\,dx$. I've mainly done this to emphasize the role of the integration by parts formula; if we just write $f(x)$ it's not as obvious that we can interpret what we're integrating as a product of two functions.

If we can integrate $f(x)$ straight away using a known result it's obviously better to do so. However, with some functions (such as $\ln x$), it isn't obvious how to integrate them. Often in these cases, integration by parts is worth a try.

Take our example of $\int \ln x\,dx$. If we set

$$\begin{cases} u = \ln x & \Rightarrow \frac{du}{dx} = \frac{1}{x} \\ \frac{dv}{dx} = 1 & \Rightarrow v = x \end{cases}$$

We obtain the result

$$\begin{aligned} \int \ln x\,dx &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C \end{aligned}$$

Again this is a fairly standard procedure, and there are not many functions other than $\ln x$ for which this is much of a problem. It can be difficult to recognise when using this method is appropriate, but when it does come up, remember to let $u = f(x)$ and $\frac{dv}{dx} = 1$.

Integrals of products of functions with cyclic derivatives

First, by "cyclic derivative" what I mean is that when you differentiate $\sin x$, for example, enough times, you end up with a multiple of $\sin x$. This doesn't happen with x^n, which is why these functions have a special category. The types of A-level integration by parts questions that usually catch people out are the ones that look like $\int e^x \sin x\,dx$. The main thing to notice here is that $\dfrac{d^2}{dx^2}(\sin x) = -\sin x$ and $\dfrac{d^2}{dx^2}(e^x) = e^x$; this implies that we need to integrate by parts twice. And we do! Because of the cyclic nature of differentiating and integrating exponential (e.g. e^x and 2^x) and sinusoidal functions (e.g. $\cos x$ and $\sin x$), it doesn't matter which way round you choose your u and $\frac{dv}{dx}$, *as long as* you are consistent. In the examples below I will always let u be the exponential function and $\frac{dv}{dx}$ be whatever is left.

Take $\int e^x \sin x\,dx$ as our first example. Choosing our u and $\frac{dv}{dx}$ appropriately, we have

$$\begin{cases} u_1 = e^x & \Rightarrow \frac{du_1}{dx} = e^x \\ \frac{dv_1}{dx} = \sin x & \Rightarrow v_1 = -\cos x \end{cases}$$

So applying the integration by parts formula we obtain

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

Now to evaluate the integral that appears here, we once again (for consistency) choose our u to be e^x, and so we have

$$\begin{cases} u_2 = e^x & \Rightarrow \frac{du_2}{dx} = e^x \\ \frac{dv_2}{dx} = \cos x & \Rightarrow v_2 = \sin x \end{cases}$$

And this gives us

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

It's at this point that many people notice that the integral on the right-hand side is the same as the one on the left-hand side, and give up. We're going around in circles, right? Well not quite! If you saw the equation $x = 2 - x$, you'd probably take the x on the RHS over to the LHS to give $2x = 2$ and solve from there. We have the same thing here; the integral $\int e^x \sin x \, dx$ is on both sides. If we denote this by A, then this equation appears like

$$A = -e^x \cos x + e^x \sin x - A$$

So rearranging and solving for A gives us

$$A \left(= \int e^x \sin x \right) = \frac{e^x(\sin x - \cos x)}{2} + C$$

where C is just some constant.

As a more complicated example, suppose we are asked to calculate

$$\int 3^{2x} \sin 2x \, dx.$$

First notice, using the laws of exponentials and logarithms, that $3 = e^{\ln 3}$, and hence that $3^{2x} = e^{(2\ln 3)x}$. Thus, we have

$$\begin{cases} u_1 = 3^{2x} \equiv e^{(2\ln 3)x} & \Rightarrow \frac{du_1}{dx} = (2\ln 3)3^{2x} \\ \frac{dv_1}{dx} = \sin 2x & \Rightarrow v_1 = -\frac{1}{2}\cos 2x \end{cases}$$

And so using our formula we obtain

$$\begin{aligned} \int 3^{2x} \sin 2x \, dx &= -\frac{3^{2x}\cos 2x}{2} - \int (-\ln 3)3^{2x} \cos 2x \, dx \\ &= -\frac{3^{2x}\cos 2x}{2} + \ln 3 \int 3^{2x} \cos 2x \, dx \end{aligned}$$

We then need to calculate the integral on the far right-hand side. Since, last time, our u term was the exponential function, we must pick our functions similarly in this step, and so

$$\begin{cases} u_2 = 3^{2x} & \Rightarrow \frac{du_2}{dx} = (2\ln 3)3^{2x} \\ \frac{dv_2}{dx} = \cos 2x & \Rightarrow v_2 = \frac{1}{2}\sin 2x \end{cases}$$

So we find

$$\begin{aligned} \int 3^{2x} \sin 2x \, dx &= -\frac{3^{2x}\cos 2x}{2} + \ln 3 \left[\frac{3^{2x}\sin 2x}{2} - \int (\ln 3)3^{2x} \sin 2x \, dx + C' \right] \\ &= \frac{[(\ln 3)\sin 2x - \cos x]3^{2x}}{2} - (\ln 3)^2 \int 3^{2x} \sin 2x \, dx + C' \end{aligned}$$

Notice now that we have our integral on both sides of the equation. If we let

$$A = \int 3^{2x} \sin 2x \, dx$$

Then we get

$$A = \frac{[(\ln 3)\sin 2x - \cos x]3^{2x}}{2} - (\ln 3)^2 A + C'$$

Solving for I then gives us

$$A = \frac{[(\ln 3)\sin 2x - \cos x]3^{2x}}{2(1 + (\ln 3)^2)} + C$$

where C is an arbitrary constant of integration. This is our final answer!

The bit that usually throws people in these types of question is noticing when the integral is on both sides of the question. The fact that this happens makes it feel like you're going round in circles. However, as we saw, it can be done just by writing A (or indeed another character) in place of the integral and solving for it as a variable in its own right. Once you've cracked this, you've cracked the method.

Bear in mind that I used a very tricky example here. I most A-level questions, your exponential function will be a power of e rather than (in this case) 3. In fact, in most cases your integral will just be $\int e^x \sin x \, dx$ or similar, which has been covered above. There is another way of tackling this type of integral which makes the use of complex numbers. I'll probably include this method in another article at some point, but it is outside the scope of both this article and the A-level Mathematics syllabus, so it will have to wait for now!

I hope this helps. If you have any feedback or queries about the article, you can send an email to *clivenewstead@yahoo.co.uk*.