Integration by substitution

by Clive Newstead

Integration by substitution looks tough when you first encounter it. My goal in writing this article is not only to try and help you understand it, but also to try to convince you that this impression really isn't the case. Once you've learnt how to spot a few patterns, you'll (hopefully) discover that integration by substitution is one of the easier parts of the A-level syllabus, rather than one of the harder ones.

If you don't have much time or patience, skip to page 5 for a summary.

I'm going to start this article on integration by substitution by talking about the chain rule. The reasons for this might not be immediately obvious, but integration by substitution can be seen as the integration counterpart to the chain rule; similarly, you can think about the chain rule as 'differentiation by substitution' − they are analogues of each other.

Say we have $y = h(x)$ as a function of x, but it is a fairly complicated function and we find that it is easier to write y as a 'function of a function'. That is, we can write $y = g(f(x))$ where g and f are two other functions. Then we can make a substitution $u = f(x)$ so that $y = g(u)$. For example, if you have $y = (2x-3)^2$, you could set $u = f(x) = 2x - 3$ and $y = g(u) = u^2$. Then we can apply the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = g'(f(x))f'(x) \tag{1}$$

to differentiate y with respect to x. So using the above example of $y = (2x-3)^2$, we have $u = 2x - 3$ and $y = u^2$, so $\frac{dy}{du} = 2u = 2(2x-3)$ and $\frac{du}{dx} = 2$, and hence

$$\frac{dy}{dx} = 2(2x-3) \times 2 = 4(2x-3)$$

Now there are two main uses that this rule can bring us with regard to integrating stuff:

1. When integrating a function of a function multiplied by the derivative of the inner function
2. When we want to transform an integral into something we recognise

Function of a function

Sometimes you might come across an integral in the form

$$\int f(g(x))g'(x)\,dx$$

This is similar to what we had in equation (1), except we don't assume that f is anything's derivative. An example of such an integral would be

$$\int 2x\sin(x^2-1)\,dx$$

Notice that $2x$ is the derivative of $x^2 - 1$, which is the 'inner function'. This is typical of an integral for which substitution needs to be used to evaluate it.

If we were to make the substitution $\boxed{u = g(x)}$, then we would have $\dfrac{du}{dx} = g'(x)$. Hence, with a little abuse of notation, we have $\boxed{dx = \dfrac{1}{g'(x)}du}$. We can implement these two substitutions (boxed) to obtain

$$\int f(g(x))g'(x)\,dx = \int f(u)g'(x)\frac{1}{g'(x)}du = \int f(u)\,du \tag{2}$$

This will usually be simpler to integrate.

Using our above example with the substitution $u = x^2 - 1$, we find that $dx = \dfrac{1}{2x}du$, so we get

$$\begin{aligned}\int 2x\sin(x^2-1)\,dx &= \int \sin u\,du \\ &= -\cos u + C \\ &= -\cos(x^2-1) + C\end{aligned}$$

Which is a lot less scary a result than it could have been!

One particular result which is very useful in integration can be derived using integration by substitution. Say we have

$$\int \frac{f'(x)}{f(x)}\,dx$$

This is in the form described above: it is the integral of a function of a function, i.e. $\dfrac{1}{f(x)}$, multiplied by the derivative of the inner function, $f'(x)$. So, we can make the substitution $u = f(x)$ to get $dx = \dfrac{1}{f'(x)}du$, and substituting in for $f(x)$ and dx, we obtain

$$\begin{aligned}\int \frac{f'(x)}{f(x)}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|f(x)| + C \end{aligned} \tag{3}$$

This can let us evaluate integrals which look very scary. Take, for example, the integral

$$I = \int \frac{\cos(2x) + x + 3}{\sin(2x) + x^2 + 6x + 9}\,dx$$

It looks pretty nasty. However, the derivative of the denominator of the fraction is $2\cos(2x) + 2x + 6$, which is exactly double what we have on the numerator of the fraction. Thus, if $f(x) = \sin(2x) + x^2 + 6x + 9$, we can write this integral as

$$\int \frac{1}{2}\frac{f'(x)}{f(x)}\,dx$$

So we can use our result from equation (3), giving that $I = \dfrac{1}{2}\ln\left|\sin(2x) + x^2 + 6x + 9\right| + C$.

Transformations of integrals

We can also use integration by substitution to transform an integral from a form we are unfamiliar with into a form we are familiar with. Take the integral

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

as an example. This isn't the integral of something simple like $\frac{1}{x^2}$ so we can't use the familiar result for this. It's not the integral of something in the form $f'(x)/f(x)$, since what's on the numerator isn't the derivative of what's on the denominator. There's also no obvious way of putting it in the form $f(g(x))g'(x)$; so what can we do?

For integrals that contain $1+x^2$ and $1-x^2$ (and particularly $\sqrt{1-x^2}$) we find that trigonometric substitutions are very useful. You should know the following two trig identities:

$$\sin^2\theta + \cos^2\theta \equiv 1, \qquad \tan^2\theta + 1 \equiv \sec^2\theta$$

These are what we will use to evaluate this integral. Rearranging the first identity, we find that $\cos\theta = \sqrt{1-\sin^2\theta}$. Returning to our above integral, this might prove useful; notice that if we let $x = \sin\theta$ then the bottom of the fraction simply becomes $\cos\theta$... this looks promising.

So we can try the substitution. If $x = \sin\theta$ then $\frac{dx}{d\theta} = \cos\theta$, so $dx = \cos\theta d\theta$. So substituting for x and dx we obtain

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-\sin^2\theta}} \times \cos\theta d\theta \\ &= \int \frac{\cos\theta}{\cos\theta}\,d\theta \\ &= \int 1\,d\theta = \theta + C \\ &= \sin^{-1}x + C\end{aligned}$$

As a general rule, here is a table of forms of integrals and the substitutions you should make to solve them:

Integral	Substitution for x	Substitution for dx	Resulting integral
$\int \frac{1}{a+bx^2}\,dx$	$\sqrt{b}x = \sqrt{a}\tan\theta$	$dx = \sqrt{\frac{a}{b}}\sec^2\theta d\theta$	$\int \frac{1}{\sqrt{ab}}\,d\theta$
$\int \frac{1}{\sqrt{a-bx^2}}\,dx$	$\sqrt{b}x = \sqrt{a}\sin\theta$	$dx = \sqrt{\frac{a}{b}}\cos\theta d\theta$	$\int \frac{1}{\sqrt{b}}\,d\theta$
$\int \frac{1}{\sqrt{ax+b}}\,dx$	$u^2 = ax+b$	$dx = \frac{2u}{a}du$	$\int \frac{2}{a}\,du$
$\int x(ax+b)^n\,dx$	$u = ax+b$	$dx = \frac{1}{a}du$	$\int \frac{1}{a^2}\left(u^{n+1} - bu^n\right)du$

These substitutions are very powerful; however, seeing $\sqrt{1-x^2}$ doesn't mean that you should immediately use a trig substitution. It is always best to make sure that a substitution like in the previous section isn't more appropriate. Say for example that you have $\int \frac{x}{\sqrt{1-x^2}}\,dx$.

In this case, it would be simpler to make the substitution $u = 1 - x^2$ rather than $x = \sin\theta$, since this *is* a function of a function multiplied by the derivative of the inner function. We could write $f(u) = \dfrac{1}{\sqrt{u}}$ and $g(x) = 1 - x^2$ so $g'(x) = -2x$, then using equation (2) we have

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int -\frac{1}{2}f(g(x))g'(x)\,dx = \int -\frac{1}{2}\frac{1}{\sqrt{u}}\,du$$

which is in a form we can integrate straight away.

Limits

When we make a substitution, we have to be careful with what happens to the limits of an integral. Say, for example, we have

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

The limits on the integral are useful if we can evaluate the integral directly, but they don't give us much information if we need to make a substitution. It would be more instructive to write

$$\int_{x=0}^{x=1} \frac{1}{\sqrt{1-x^2}}\,dx$$

Because then we can see that it is x, and not any substituted variable, that takes the values 0 and 1 at the end points of our region of integration. If we make the substitution $x = \sin\theta$, as above, then these limits change. If $x = 1$ then $\theta = \sin^{-1}(1) = \frac{\pi}{2}$, and if $x = 0$ then $\theta = 0$. So, using our result, we have

$$\begin{aligned}\int_{x=0}^{x=1} \frac{1}{\sqrt{1-x^2}}\,dx &= \int_{\theta=0}^{\theta=\pi/2} 1\,d\theta \\ &= [\theta]_{\theta=0}^{\theta=\pi/2} \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2}\end{aligned}$$

Notice that changing the limits meant that we never needed to make the reverse substitution at the end like we did when evaluating the integral without any limits there. In fact, we could have gotten away without changing the limits at all; as long as we make the reverse substitution we are in business:

$$\begin{aligned}\int_{x=0}^{x=1} \frac{1}{\sqrt{1-x^2}}\,dx &= \int_{x=0}^{x=1} 1\,d\theta \\ &= [\theta]_{x=0}^{x=1} \\ &= [\sin^{-1} x]_{x=0}^{x=1} \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2}\end{aligned}$$

So whether you choose to change the limits and evaluate straight away, or leave the limits and make a reverse substitution before evaluating, is a choice you should make based on which you think will be easiest.

Summary

Most of this article has been about how to spot what kinds of substitutions to make. Don't be angry at me when I say that in an A-level C3/4 exam the substitutions are usually (but not always) given to you, so some of what you have just read is, strictly speaking, extracurricular. Nevertheless, knowing why the substitutions work is what makes using them feel so much easier. Indeed, if you're taking Further Maths at A-level then you'll need to know the substitutions for the Further Pure modules. In any case, I'm going to summarise what steps you should take when integrating by substitution:

1. Decide the substitution you're going to use. This might be in the form $u = f(x)$ or $x = g(u)$
2. Find what dx becomes. If $u = f(x)$ then $dx = \frac{1}{f'(x)} du$, and if $x = g(u)$ then $dx = g'(u) du$ – the difference is important!
3. If your substitution is $u = f(x)$, then replace all occurrences of $f(x)$ by u; if your substitution is $x = g(u)$ then replace all occurrences of x by $g(u)$
4. Replace all occurrences of dx by the appropriate substitution (see step 2)
5. Check your limits (if there are any)! You can either change the limits now, compute the integral and then evaluate; or leave the limits as they are, compute the integral, substitute back and then evaluate

I hope this helps. If you have any feedback or queries about the article, you can send an email to *clivenewstead@yahoo.co.uk*.